AIAA JOURNAL
Vol. 31, No. 6, June 1993

Design of Frames Against Buckling Using a Rayleigh Quotient Approximation

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A Rayleigh quotient approximation is applied to the design of structures while guarding against elastic instability. It approximates buckling eigenvalues by separately estimating the modal strain energy due to the linear and geometric stiffness of the structure. Previously used during structural optimization for the fundamental natural eigenvalue, the Rayleigh quotient approximation is derived for the buckling design problem for the first time. The critical buckling load is found by solving the eigenvalue problem that arises by considering the geometric nonlinearity of the deforming structure. Rayleigh's principle is used to justify the choice of intermediate design variables for approximating terms in the Rayleigh quotient. A truss model illustrates the importance of the design space chosen for approximating the modal strain energy. A beam-column, two plane frames, and a space frame are used to verify the formulation. Special attention is paid to difficult bimodal optima.

Introduction

ARBITRARILY large deformations and even catastrophic failure may occur for structures with certain geometry and loading. The phenomenon is termed buckling when sudden large deformations due to geometric nonlinearity result in dynamic instability. If the structure is loaded in a quasisteady manner, linear behavior is often observed nearly until the critical load at which buckling occurs. During the large deformations of buckling, the applied but as yet unknown load is considered to be constant in direction and magnitude. Thus, the nonlinear stiffness may be considered the addition of the usual linear stiffness (constant terms that multiply the displacements linearly) and non-linear terms called the geometric or differential stiffness. When only terms linear with respect to internal loads are retained in the latter, an eigenvalue problem may be solved to determine the magnitude of the buckling load of a given direction. The design goal is to find the "best" distribution of material that avoids this global buckling.

The buckling design problem may be defined as finding the minimum weight structure that satisfies a prescribed buckling load. Alternatively, it may be to maximize the buckling load for a structure with a given volume, mass, or weight. Other performance criteria, such as strength and stiffness, may be included as well. Tadjbakhsh and Keller[1] were the first to present the optimal shape for the strongest column of a given length and volume for several boundary conditions, a problem first posed by Lagrange in 1773. Their results for clamped-clamped and clamped-hinged columns stood unchallenged until 1977 when Olhoff and Rasmussen[2] recognized that these cases were bimodal optima. In fact, the previous results were not optimal because the second buckling mode crossed the first and became critical at a lower load level. The true optimal design is bimodal, i.e., the critical load is governed by a repeated eigenvalue. Others developed algorithms based on the finite element method to solve the buckling design problem.[3] Szyszkowski et al.[4] generalized the bimodal optimality criteria[2] for the finite element method.

Recently, Canfield developed a Rayleigh quotient approximation (RQA) for eigenvalues during structural design.[5] Although the RQA was first used for natural frequencies, the concept applies to buckling eigenvalues as well. The current work demonstrates the RQA in buckling design problems for the first time. Rayleigh's principle is followed as a guideline to selecting the best intermediate variables for approximating the modal strain energies of the lower modes. This important step has been overlooked by others in the literature who have applied the RQA.[6-8] From the derivation it is clear that the RQA only offers some benefit to buckling problems when multiple load paths exist, i.e., for an indeterminate structure. Otherwise, the RQA reduces to the same equation as a direct approximation of the eigenvalue. Its performance for troublesome bimodal optima is of particular interest. Repeated eigenvalues are not continuously differentiable, potentially crippling gradient-based optimization methods. In fact, this study is the first to examine the RQA for more than the fundamental eigenvalue.

The RQA requires derivatives of the modal strain energies appearing in the Rayleigh quotient. Thus, sensitivity of the element's stiffness, geometric stiffness, and mass to its cross-sectional properties must be calculated. Martin's derivation of the finite element differential stiffness matrix is followed[9] for the element matrices of a frame. The element used is the superposition of a two-noded linear rod and torsion element with two classical Euler-Bernoulli beam elements (no shear deformation) that use Hermite cubic interpolation. The displacements of the two beam elements are mutually orthogonal and the plane of each beam is assumed to lie in one of the principal directions (no products of inertia). The design variables used in the following derivation of eigenvalue approximations are cross-sectional properties (areas and moments of inertia). When cross-sectional dimensions control the redesign, they are used to reconstruct the cross-sectional properties, and appropriate chain rules are used for the gradients.[10]

Theoretical Formulation

The buckling problem may be described as determining the critical load associated with structural instability for a prescribed loading configuration. The critical load may be found by solving the eigenvalue problem arising from the geometric nonlinearity in the field equations. Two sources of nonlinear behavior are common in solid mechanics: material nonlinearity due to nonlinear constitutive equations and geometric nonlinearity. The latter, considered here, is attributable to two sources: the strain-displacement relations and the equilibrium equations.

Geometric nonlinearity is introduced by employing the nonlinear strain-displacement equation. Although the field equations can be developed for continuous systems directly, they

Received Feb. 10, 1992; presented as Paper 92-2249 at the AIAA/ASME/ASCE/AHS/ASC 33rd Structures, Structural Dynamics, and Materials Conference, Dallas, TX, April 13-15, 1992; revision received Sept. 21, 1992; accepted for publication Sept. 24, 1992.

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are presented here for systems with finite degrees of freedom. The nonlinear terms can be separated from the linear ones in the discretized system equations

$$[K^0 + K^G(\sigma)]\phi = P \tag{1}$$

where K^0 is the linear stiffness, and K^G is the nonlinear stiffness—a function of the stress state σ and hence of the internal loads F. The system degrees of freedom are represented by the vector ϕ of Rayleigh-Ritz coefficients (nodal displacements in the finite element method). The direction and relative magnitudes of the applied load are represented by the vector P. Assuming that the internal stresses can be obtained from the linear equations involving K^0 only and that they remain constant during the transition to the buckled state (in keeping with the assumption that the externally applied load remains constant in magnitude and direction), the eigenvalue problem evolves as

$$|K^0 - \lambda K^1(F)| = 0 \tag{2}$$

where K^1 is the geometric stiffness containing only terms linear in the internal forces. Introducing a change of sign for the first-order geometric stiffness K^1 makes it a function of the compressive forces in anticipation of forming the Rayleigh quotient.

The current study demonstrates the usefulness of RQA beyond the fundamental eigenvalue for a natural frequency. In their landmark study, Olhoff and Rasmussen showed that bimodal optimality criteria lead to the correct optimal design of columns.[2] According to Szyszkowski et al. "gradient-based iterative methods cannot be used" for multimodal optimization.[4] Finite element examples from that paper will be used to show that gradient-based mathematical programming can handle bimodal optima when the RQA is used for both modes. Thus, multimodal optimization can be subsumed under general mathematic programming to facilitate the incorporation of other constraints.

First-Order Approximations

An appropriate set of intermediate variables for constructing the Taylor series approximation to a function is crucial in achieving accurate approximations.[11] Only first-order approximations are considered here for the sake of efficient sensitivity computations. When the n variables of the design vector x are used to construct a first-order Taylor series for a function $g(x)$ the approximation

$$g_L(x) = g(x_0) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\bigg|_{x_0} (x_i - x_{0i}) \tag{3}$$

is linear.

If the reciprocal of each variable $v_i = 1/x_i$ is defined as an intermediate design variable, a linear approximation with respect to the reciprocal variables can be constructed as in Eq. (3). Substituting the original variables for the intermediate reciprocal variables results in

$$g_R(x) = g(x_0) - \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} x_i^2\right)\bigg|_{x_0} \left(\frac{1}{x_i} - \frac{1}{x_{0i}}\right) \tag{4}$$

called the reciprocal approximation.

Subtracting Eq. (4) from Eq. (3), one observes that the sign of the derivative determines for each variable which form contributes more in the positive sense to the approximation of g. The so-called conservative approximation[12]

$$g_C(x) = g(x_0) + \sum_{i=1}^{n} G_i \frac{\partial g}{\partial x_i}\bigg|_{x_0} (x_i - x_{0i}) \tag{5}$$

where

$$G_i = 1 \qquad \text{if } \frac{\partial g}{\partial x_i} x_i \bigg|_{x_0} \geq 0$$

$$= x_{0i}/x_i \qquad \text{otherwise}$$

is an overestimate relative to the previous two approximations. If g represents a constraint function which must remain negative to be feasible, overestimating the constraint is conservative.

Role of Rayleigh Quotient Approximation in Approximate Subproblem

Iterative, nonlinear mathematical programming methods abound for solving the optimization problem described. Most require derivatives of the objective and constraint functions with respect to the design variables. The number of iterations are typically proportional to the number of design variables. Because of the numerical expense of calculating response quantities and their derivatives, numerous iterations are not tolerable for any but academic problems. The prevalent approach is to form approximations to the response quantities of interest at each iteration and solve the resulting approximate, but explicit, subproblem using the nonlinear programming methods. Typically, only a dozen or so complete response and sensitivity analyses are needed for convergence to a nearly optimum design, independent of the number of design variables. The success of this approach depends on high quality approximations of the response quantities.

Derivation of the RQA originates with Rayleigh's energy method[13] which employs an estimate of the eigenvector ϕ to approximate the eigenvalue. In terms of the system stiffness matrices K^0 and K^1, the Rayleigh quotient is

$$\lambda = \frac{\phi^T K^0 \phi}{\phi^T K^1 \phi} = \frac{U}{G} \tag{6}$$

where U represents the linear modal strain energy and G the nonlinear (geometric stiffness) strain energy of the buckled state. They are the sum of modal energies of all the structural components in the system. We choose the buckling mode from the eigenanalysis as an estimate during the solution of the approximate subproblem. Thus, during redesign, the Rayleigh quotient [Eq. (6)] only requires an estimate of the system matrices appearing in the numerator and denominator. In fact, the new linear stiffness matrix is often simple to calculate, because it is typically linear in the intermediate design variables. Unlike the natural frequency problem, the denominator can only be estimated for buckling using the internal load sensitivity. Once the estimated system matrices are pre- and postmultiplied by the constant eigenvector, the RQA represents an approximation to the modal energies appearing in its numerator and denominator. As the design vector changes, Eq. (6) can be used to estimate the new eigenvalue where only K^0 and K^1 are considered functions of the design variables. Essentially, the eigenvector from the eigenanalysis is a Ritz vector during optimization of the approximate subproblem.

As previously mentioned, the appropriate design space is paramount. Moreover, Rayleigh's principle[13] guarantees that the Rayleigh quotient using an estimated eigenvector will overestimate the fundamental eigenvalue. Overestimating the critical load during buckling design leads to elastic instability—precisely what the designer intends to avoid. To achieve a more conservative estimate an appropriate first-order approximation can be selected for the modal strain energies appearing in Eq. (6), thereby compensating for the known error. Whereas the mass matrix of a perturbed design can be constructed easily for most vibration analyses, the geometric stiffness must be approximated using the internal load sensitivity. In the former case, substituting the exact perturbed modes into Rayleigh's quotient would yield the corresponding eigenvalues identically. In contrast, Rayleigh's quotient for buckling would still be approximate unless the geometric stiffness were completely regenerated. Assuming the exact perturbed geometric stiffness were known, then according to Rayleigh's principle any perturbations in the mode due to design variable changes act to reduce the fundamental eigenvalue relative to its approximation. Thus the Rayleigh quotient approximation is formed according to the criteria used for Eq. (5) as

$$\lambda_{\text{RQA}} = \frac{U_R}{G_C} = \frac{U_0 + \sum_{i=1}^{n} \frac{x_{0i}}{x_i} u_i' (x_i - x_{0i})}{G_0 + \sum_{i=1}^{n} G_i g_i' (x_i - x_{0i})} \tag{7}$$

where

$$u_i' = \phi^T \frac{\partial K^0}{\partial x_i} \phi \bigg|_{x_0} \tag{8}$$

$$g_i' = \phi^T \frac{\partial K^1}{\partial x_i} \phi \bigg|_{x_0} \tag{9}$$

$$\begin{aligned} G_i &= 1 && \text{if } g_i' \ge 0 \\ &= x_{0i}/x_i && \text{otherwise} \end{aligned} \tag{10}$$

U_0 is the linear modal strain energy and G_0 the geometric modal strain energy of the unperturbed design. Since the geometric stiffness depends linearly on the internal loads, the chain rule is applied to Eq. (9)

$$\phi^T \frac{\partial K^1}{\partial x_i} \phi = \sum_{k=1}^{N} \frac{\partial f_k}{\partial x_i} \phi^T \frac{\partial K^1}{\partial f_k} \phi \tag{11}$$

so each derivative involves a summation over all N elements. For indeterminate finite element structures the summation can be carried out efficiently with element matrices in local coordinates. For determinate structures $(\partial f_k/\partial x_i) \equiv 0$ by definition, and the geometric modal strain energy is approximated as a constant. In this case the RQA reduces to the reciprocal approximation to the eigenvalue, $\lambda_{RQA} = U_R/G_0 = \lambda_R$. Considering that the linear strain energy derivative u_i' is positive (since K^0 is positive definite and linear in x), the conservative approximation for a buckling eigenvalue of a determinate structure also reduces to the reciprocal approximation when cross-sectional properties are the intermediate design variables. In general, the geometric strain energy derivative g_i' is sign indefinite. For nonpositive definite geometric stiffness the critical buckling load (lowest positive eigenvalue) does not correspond to the fundamental eigenvalue, which is negative. In this case we choose to be conservative without justification by Rayleigh's principle. Nevertheless, the RQA as derived proved to suffice in the subsequent examples with indeterminate structures.

Internal loads for the finite element method can be expressed as

$$f_i = k_i^0 \hat{u}_i, \qquad \forall i \in \{1, \ldots, N\} \tag{12}$$

where, for the ith element, f_i is the internal load vector, k_i^0 the local element linear stiffness, and u_i the vector of element displacements in local coordinates due to the applied load P.

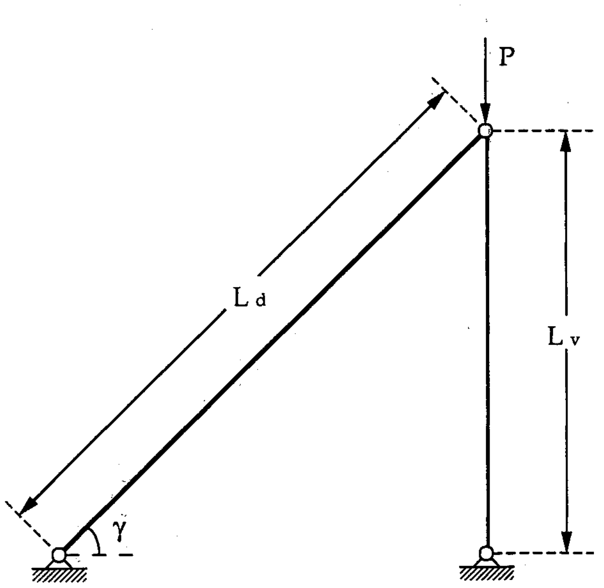

Fig. 1 Two member truss.

By differentiating Eq. (12) the geometric stiffness sensitivity involves

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial k_i^0}{\partial x_j} \hat{u}_i + k_i^0 \frac{\partial \hat{u}_i}{\partial x_j}, \qquad \forall i \in \{1, \ldots, N\}, \quad j \in \{1, \ldots, n\} \tag{13}$$

and the last term requires differentiation of the system equation for the linear static solution $P = K^0 u$

$$\frac{\partial u}{\partial x_j} = -(K^0)^{-1} \frac{\partial K^0}{\partial x_j} u \tag{14}$$

for each design variable. The Eqs. (11), (13), and (14) are substituted into Eq. (9) to obtain the geometric modal strain energy gradient.

The derivative of an approximate eigenvalue estimated by Eq. (7) is

$$\frac{\partial \lambda_{RQA}}{\partial x_i} = \frac{u_i' \left(\dfrac{x_{0i}}{x_i} \right)^2 - \lambda_{RQA} G_i'}{G_0} \tag{15}$$

where

$$\begin{aligned} G_i' &= g_i' && \text{if } g_i' \ge 0 \\ &= g_i' \left(\frac{x_{0i}}{x_i} \right)^2 && \text{otherwise} \end{aligned} \tag{16}$$

and u_i' and g_i' are the linear and geometric modal strain energy sensitivities, respectively, from Eqs. (8) and (9). An important characteristic of Eq. (15) for indeterminate structures is that the derivative can potentially change sign to better follow the nonlinear response as the design variables change. This trait does not hold for any of the approximations in Eqs. (3–5). Once the explicit and approximate subproblem is defined using Eqs. (7–10), (15), and (16), it may be solved by an optimizer of choice.

Results

Five problems from the literature were identified for validation of the design capability. Two possible problem statements were posed. One was to minimize the structural weight subject to a limit on the critical buckling load. The second was to maximize the critical buckling load for a given volume of material. The optimum solution for either problem may be bimodal. The former case requires constraining both of the two lowest positive eigenvalues. Otherwise, so-called "mode-switching" will cause oscillations in the design iterations as the second mode becomes critical at each cycle. In the latter case, the second eigenvalue is required to be no less than the first. Mathematically, the design problem is

$$\max[\lambda_1(x)] \tag{17}$$

subject to

$$V(x) = V_0 \tag{18}$$

where λ_1 is the lowest positive eigenvalue, V the total volume of the structure, and V_0 the given amount of material. Numerically, after each eigenanalysis, a nonlinear optimization routine is used to solve the approximate subproblem

$$\max[\lambda_{RQA}^{(1)}(x)] \tag{19}$$

subject to

$$\lambda_{RQA}^{(j)}(x) \ge \lambda_{RQA}^{(1)}(x), \qquad \forall j \in \{2, \ldots, J\} \tag{20a}$$

$$\underline{x}_i \le x_i \le \bar{x}_i, \qquad \forall i \in \{1, \ldots, n\} \tag{20b}$$

$$V(x) = V_0 \tag{20c}$$

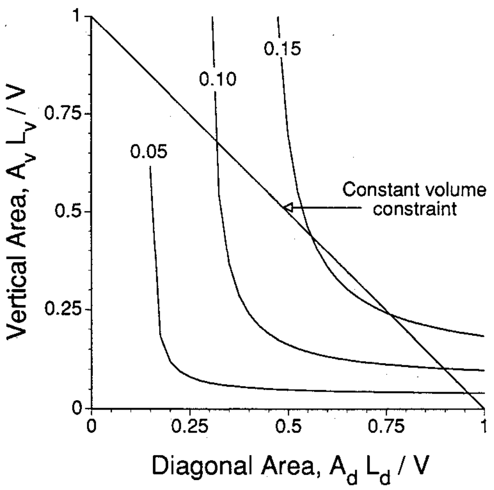

Fig. 2 Buckling load contours.

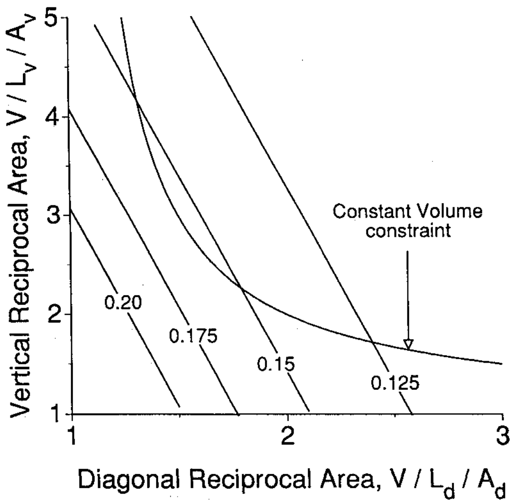

Fig. 3 Buckling load contours in reciprocal design space.

where $\lambda_{RQA}^{(j)}$ are the approximations of the modes with the J lowest positive eigenvalues from the last structural analysis. Side constraints are also placed on the design variables based on physical considerations and possibly from move limits to ensure accuracy of the approximations. Although not stated explicitly in the preceding formulation, the frame cross-sectional properties (area and moment of inertia) were used as intermediate design variables for functions involving the element stiffness.[10] The actual design variables determined by the optimization routine were the cross-sectional dimensions specified in each problem description to follow.

Two Member Truss

A simple truss from Ref. 9 is reproduced in Fig. 1. The cross-sectional areas of the diagonal and vertical truss members are A_d and A_v, respectively. The problem can be made nondimensional by defining $\alpha_d \equiv A_d L_d / V_0$ and $\alpha_v \equiv A_v L_v / V_0$ so that

$$\mathcal{P} \equiv \frac{P_{\text{crit}} L_d}{E V_0} = \frac{\alpha_d \sin\gamma \cos^2\gamma}{1 + (\alpha_d/\alpha_v)\sin^4\gamma}$$

Figure 2 is a nondimensional contour plot for $\gamma = 45$ deg of the normalized critical load $\mathcal{P}$. The feasible design space lies along the line marked "constant volume constraint" for which $\alpha_d + \alpha_v = 1$. This plot can be used to graphically locate the optimum where the level curve for $\mathcal{P}$ is tangent to the constraint line.

If the parameters were normalized by some initial volume, a similar plot could be used for the minimum volume design for a given buckling load. In this case, one of the level curves for $\mathcal{P}$ represents the constraint, whereas the constant volume curve becomes a series of parallel lines for the level curves of volume. Also, the axis ranges might extend beyond unity. Again the optimum would occur where the critical load curve was tangent to one of the volume level curves. Analytic solution of the Kuhn-Tucker optimality conditions for this simple example leads to the condition $\alpha_v/\alpha_d = \sin^2\gamma$ for both design problems. Substitution into the appropriate constraint condition yields the optimum design variable values. For the constant volume constraint and $\gamma = 45$ deg, $\alpha = 1/3$, $\alpha_d = 2/3$. If the optimum were found by solving the approximate subproblem using Eq. (3) at each design iteration, convergence would be controlled by the move limits, since the design would always move as far as possible along the constraint line.

Next consider the same contour plot constructed in the reciprocal design variable space (Fig. 3). Although the critical load is still a nonlinear function of the reciprocal variables, each level curve is linear. (The level curves would be evenly spaced for a linear function.) As a result, the numeric solution for the maximum buckling load design using the RQA (or reciprocal or conservative approximations) requires only a single iteration! Although the approximation is not exact, the direction of its gradient is always correct, so only one iteration is required to find the point of tangency to the nonlinear, but explicit, volume constraint. On the other hand, solution of the minimum volume problem requires additional iterations because the approximation is not exact. In fact, two iterations were required to reduce the volume by 7% for a buckling constraint $\mathcal{P} = 0.1464$, initial values $\alpha_d = 0.5858$ and $\alpha_v = 0.4142$, and no move limits.

Because the two bar truss is determinate, an RQA is equivalent to reciprocal and conservative approximations. To illustrate the difference a third bar was added to the truss, its base pinned halfway between the other two members (Fig. 4 inset). For unit height and elastic modulus, the volume was optimized subject to a buckling constraint for a unit load. To examine the quality of approximations, initial areas were chosen so as to force the design to move through large distances in the design space to reach the optimum. The diagonal, middle, and vertical areas were assigned 2.0, 6.0, and 0.1 units, respectively. Minimum area was 0.1 unit and no move limits were imposed. Although close to the optimum volume, the initial design was grossly infeasible. In fact, reaching the optimum involved transferring all material from the middle bar to effectively eliminate it. The final design was $\{A_d, A_m, A_v\} = \{4.219, 0.1, 3.031\}$ and iteration histories using the various approximations appear in Fig. 4. The lower curves represent

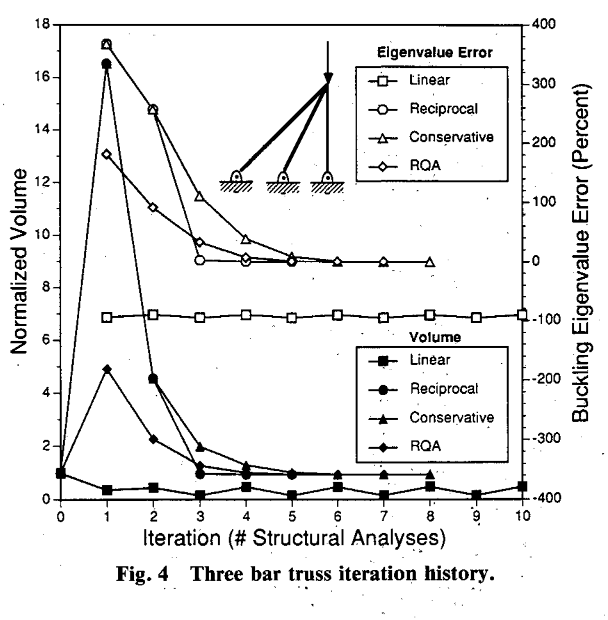

Fig. 4 Three bar truss iteration history.

volume and the upper curves the buckling eigenvalue approximation error. The linear approximation never converged. At each cycle the single most effective variable was used to satisfy the constraint and the others were set to minimum. The approximate subproblem was linear, hence its solution was always at a vertex in the design space. The reciprocal and conservative approximations were initially the same because the buckling derivatives all had the same sign. In the third iteration the derivative with respect to the middle bar's area changed sign, so that it was set to minimum using the reciprocal approximation. The RQA was the most accurate when large changes in the design were made. The level of accuracy depended on how large a design change was made, as well as where in the design space the structure was analyzed.

Clamped-Clamped Beam Column

The bimodal optimum of Olhoff and Rasmussen[2] for the clamped column was duplicated using the RQA approach. A 100 frame element representation of a beam column was used, whereas Ref. 2 employed a finite difference formulation to iteratively solve the nonlinear integro-differential eigenvalue problem of the continuous system. They assumed the moment

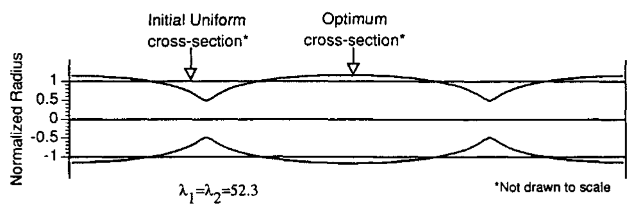

Fig. 5 Clamped beam column optimum profile.

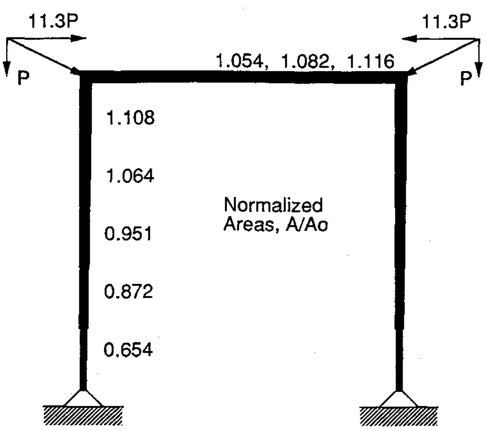

Fig. 6 Portal frame optimum design.

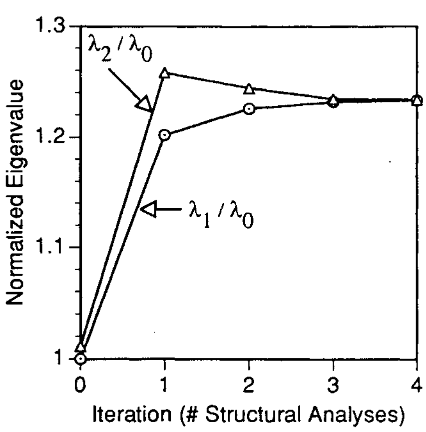

Fig. 7 Portal frame iteration history.

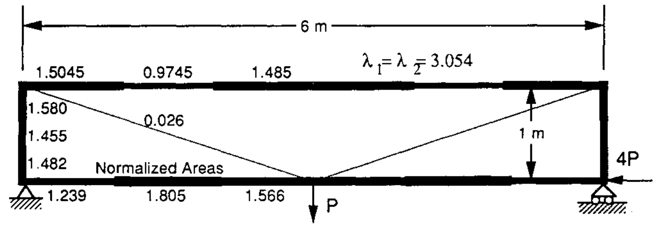

Fig. 8 Two bay frame optimum design.

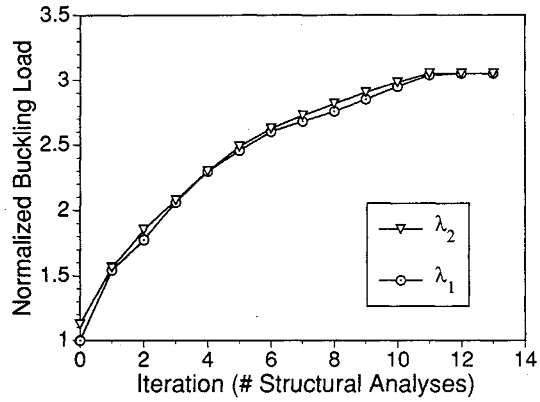

Fig. 9 Two bay frame iteration history.

of inertia was proportional to the square of the column's cross-sectional area. Hence, the radius of a solid circular cross section served as the design variable for the current compatible finite element model. The current method converged after four iterations to $\lambda_1 = 52.264$ and $\lambda_2 = 52.294$, compared to $\lambda_1 = \lambda_2 = 52.356$ for the case in Ref. 2 with no lower bound on the cross-sectional area. The radius as a function of span found using the current approach (Fig. 5) corresponds to Olhoff and Rasmussen's result.

Portal Frame

A fifteen member square portal frame[4] was modeled by dividing the posts and lintel into five frame elements each. The bimodal optimum was found in four iterations using the RQA compared to 13 iterations in Ref. 4. The ratio of the final double eigenvalue to the initial fundamental eigenvalue was 1.234 in both cases. The width of each square cross section was taken as a design variable, and the ratios of the final to initial uniform widths are displayed for half of the symmetric model in Fig. 6. An iteration history of the two lowest eigenvalues is shown in Fig. 7. The RQA produced results identical to the reciprocal approximation due to the low degree of indeterminacy. Move limits were not applied. In contrast the linear approximation required that move limits be tightened from 30% to 1% to converge in 16 iterations (Table 1). It is interesting that all approximations were nonconservative (overestimated the eigenvalue) for this problem.

Two Bay Frame

The most difficult example in Ref. 4 appeared to be a two bay frame (Fig. 8) having its two most critical eigenvalues closely spaced. The critical eigenvalue of the nominal design was reproduced by selecting an appropriate elastic modulus (203.5 GPa). The height of each solid rectangular cross section (initially 4 cm by 4 cm) was treated as a design variable. Analyzing the final design from Ref. 4 reproduced the same maximum critical load (3.02 times the load for the uniform design); however, the analysis revealed the volume had increased by 5% and the design was not bimodal as claimed. To reproduce the bimodal optimum the ratio of horizontal to vertical loads was reduced from 10:1 to 4:1. For this load case

Table 1 Approximation errors for portal frame

	Critical eigenvalue			Approximation error, %		
Iteration	Linear	Reciprocal	Conservative	Linear	Reciprocal	Conservative
0	0.8100	0.8100	0.8100	—	—	—
1	0.5644	0.9737	0.9676	151.48	1.189	0.248
2	0.8039	0.9931	0.9952	4.87	1.212	0.220
3	0.8965	0.9982	0.9991	2.69	0.212	0.015
4	0.9318	0.9991	0.9991	0.72	0.031	0.010
16	0.9987	—	—	0.04	—	—

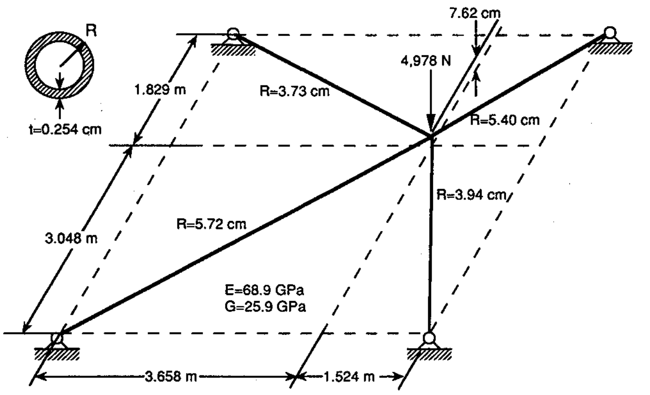

Fig. 10 Four bar frame optimum design.

the ratio of the final bimodal to the initial uniform critical load was 3.38. With this modification in loads, the bimodal solution was found in 13 iterations (Fig. 9) without move limits, as compared to more than 25 iterations in Ref. 4. The reciprocal and conservative approximations also converged to the same optimum in 13 iterations. The linear approximation did not converge to the optimum within 20 iterations for any set of move limits. The buckling load was increased by 3.051 while precisely maintaining the volume constraint. The optimum to initial uniform height ratios are shown for half of the symmetric model in Fig. 8.

Space Frame

A space frame modeled with four beam elements pinned at the base and clamped at the apex was adapted from Ref. 14 (Fig. 10). Each leg is a circular tube, modeled as a frame instead of a truss. The thickness and radius of each tube served as design variables, initially 0.508 cm and 2.54 cm with minimum values of 0.254 cm and 1.27 cm, respectively. When maximized for buckling load, all thicknesses went to minimum. The final load and radii are shown in Fig. 10. Convergence to the bimodal optimum, achieved in 8–12 iterations, was sensitive to move limits. Comparisons of RQA to the conservative approximation was also move limit dependent. Generally, the RQA progressed more quickly toward the optimum for move limits greater than 30–40%, but was no more accurate than the conservative approximation for smaller move limits. Volume minimization subject to an 890 N load gave a trimodal optimum, and convergence was even more difficult.

Conclusions

The eigenvalue problem for structural instability and its Rayleigh quotient approximation were derived in general, and the geometric stiffness sensitivity was developed for space frame elements in particular. The derivation involved two crucial steps. First, each eigenvector corresponding to the most critical eigenvalues from the structural analysis was used as a Ritz vector to form Rayleigh's quotient during redesign. Modal strain energy was thereby approximated as an intermediate response quantity. Unlike the system mass matrix in natural vibration problems, the geometric stiffness matrix in buckling problems cannot be calculated exactly for a perturbed design vector. Thus, the denominator in Rayleigh's quotient was estimated using only a first-order approximation of the internal forces. In the second critical step, Rayleigh's principle was used to select the most appropriate type of approximation for the modal strain energies. The derivation showed that the Rayleigh quotient approximation reduces to a reciprocal approximation of the eigenvalue in the case of determinate structures. Examples from the literature were used to demonstrate the RQA and a formulation to handle multimodal optima.

A simple analytic truss example verified the choice of an intermediate design space. For an indeterminate version of this truss the RQA appeared to characterize the buckling eigenvalue over a large region of the design space better than other approximations to the eigenvalue. Near the point of approximation, however, it was in general no more accurate than reciprocal or conservative approximations. Hence for small design changes or for determinate structures the RQA offers little benefit in buckling design, although the derivation lends insight into selection of an appropriate design space even for these cases. Estimating modal energy in the denominator of Rayleigh's quotient distinguishes the RQA from the usual eigenvalue approximations. For buckling problems sensitive to internal load redistribution, accuracy of the internal force sensitivity is a limiting factor for any approximation. Estimating the geometric stiffness matrix limits the accuracy of an RQA to a buckling eigenvalue more so than for vibration eigenvalues where the perturbed mass matrix can be known exactly. Nevertheless, numeric examples demonstrated that the RQA, in conjunction with the proper formulation of the approximate subproblem, can indeed converge to bimodal optima. Smooth convergence was guaranteed by incorporating a constraint to prevent the second mode from crossing below the first. The plane frame designs converged to bimodal optima without move limits.

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